Stability of Dynamical Systems: An Overview

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Introduction

MATHEMATICAL equations representing most physical systems are not amenable to closed-form solution. In most cases, this is not as serious an obstacle to understanding the behavior of the system as may be expected, for we may relinquish the path of determining explicit solutions and take recourse to the many qualitative methods of analysis in vogue that determine the properties of solutions without explicitly solving the equations.

Of the properties of solutions, stability is perhaps the most important. This stems from three factors: First, in any physical problem, it is the stable solutions that are of interest. As engineers, we wish to be informed about the unstable solutions only so that we may elude them in design. Second, unstable solutions are extremely sensitive to errors. Compounded with the fact that no physical law is exact, this will render such solutions useless as they will turn out to be poor approximations to observed natural phenomena. Finally, although the unstable states are mathematically rigorous solutions of the differential equations, they cannot be observed in practice owing to the small disturbances that exist in the initial conditions and in the forces acting on the system; which have to be disregarded in the formulation of the problem, either because of the unwieldy equations that result or because of the sheer ignorance of the way they manifest themselves. Consequently, it is important to study the sensitivity of systems to small perturbations in their initial states and in the governing equations.

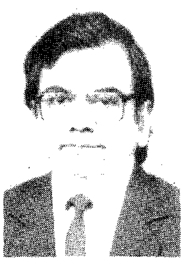
In this paper, the stability theory of dynamical systems is reviewed in a historical perspective. In writing a survey that presents a unified view of a large number of diverse results obtained by numerous scientists over centuries, one is forced to pick out the most important lines of thought and leave the details to more specialized publications. The important literature relevant to each section in this paper is so extensive that to refer to all of it here is too ambitious a task. We have confined ourselves to indicating only the main points.

What Is Stability?

The word *stable* derives from the Latin adjective[1] *stabilem*, which means "being able to stand firmly." The English word shares the same meaning as its Latin root. Surprisingly, in scientific literature, stability has no universally accepted definition. It is roughly characterized by the response of a system to small disturbances in its state. A system whose stability is to be studied is perturbed a little from its equilibrium; if the perturbed motion stays within acceptable limits of the equilibrium position, it is said to be stable; otherwise it is unstable.

This basic notion has been interpreted in several ways. Formerly, stability was defined in relation to the equilibrium states of a system of bodies. A cone standing on its base (apex) being in stable (unstable) equilibrium is an example of this concept. The inadequacy of this simplistic definition in most situations resulted in the introduction of new definitions, each in some way related to the commonsense notion of stability. The following definitions are listed to highlight the diverse interpretations of this term.

Lyapunov stability: For small values of initial disturbances, if the disturbed motion constrains itself to an arbitrarily prescribed small region of the state space, then the system is said to be stable in the sense of Lyapunov (i.s.L.).



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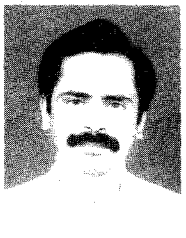

Born in 1959, S. Pradeep received his Bachelor's degree in 1981 in Aeronautical Engineering from the Indian Institute of Technology, Kharagpur. Subsequently, he moved to the Indian Institute of Science, Bangalore, where he received his Master's and Doctoral degrees in Aerospace Engineering in 1983 and 1987, respectively. Since then, he has been working as a Research Associate in the Department of Aerospace Engineering at the Indian Institute of Science. He was selected as the Jamshed R. D. Tata Research Fellow for 1985–1987. He was also the recipient of the best thesis award. His research interests include stability theory and the dynamics and control of large space structures.

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Asymptotic stability (i.s.L.): If the system is stable i.s.L. and all solutions approach zero for large values of time, then it is asymptotically stable i.s.L.

Stability in the large: A system is stable in the large if it is stable i.s.L. for all initial disturbances.

Orbital stability: The system is orbitally stable if the disturbed trajectory and the reference trajectory are arbitrarily close, with the correspondence not necessarily isochronic.

Poisson stability: The system is required to return infinitely often, arbitrarily near the equilibrium position, while the intervening oscillations are permitted to be of any magnitude.

Structural stability: A system is structurally stable if two neighboring solutions remain sufficiently close to each other as the parameters of the system change.

Finite-time stability: This differs from the aforementioned concepts in that it focuses on the behavior of trajectories that originate within an a priori fixed region in the state space over a given fixed interval of time.

Stochastic stability: The deterministic concepts of stability have their counterparts in each of the common modes of convergence of probability theory, namely, convergence in probability, convergence in the mean, and almost sure convergence.

In a recent paper, Szebehely[2] enumerated, in dictionary form, more than 50 definitions of stability.

Genesis

The concept of stability is as old as the practice of engineering. The gigantic structural monuments of early civilizations bear testimony to this. A proper foundation for stability theory was laid by Aristotle and Archimedes,[3,4] who originated two different schools of thought: Aristotle propounded the kinematic method and Archimedes the geometric method. Further progress in dynamic stability was slow until the discovery of the laws of motion.

The principle governing static stability has been known for a very long time. This principle, which Chetayev[5] attributes to Toricelli, states: "Two connected bodies cannot move of their own accord unless the motion is in concordance with the falling of their center of gravity."[5] This result of Toricelli was based on another theorem, according to which, "In a system of rigid bodies in a state of equilibrium, the center of gravity occupies the lowest relative position possible." Static stability is characterized completely by these theorems. Unfortunately, a corresponding principle to determine dynamic stability has not yet been discovered.

Linear Time-Invariant Systems

Closed-form solutions exist for linear time-invariant systems in terms of an exponential. Based on this solution, it can be shown that the necessary and sufficient condition for the solutions of the system $x' = Ax$ to be asymptotically stable is that all the characteristic roots of A have negative real parts. The difficulty in computing the characteristic roots for higher-order systems (sans a digital computer) was the primary motivation for the development of qualitative methods of stability analysis for linear time-invariant systems. This section surveys the most important of these qualitative methods.

The first notable contribution was made by Lagrange,[6] who extended Toricelli's static stability criterion to dynamics and showed that the equilibrium point of a mechanical system is stable when the force function acting on the system has a maximum at this position. He took the potential function to be quadratic; Dirichlet[7] extended it for a general potential function. Later, Painlevé[8] and Winter[9] weakened the hypothesis of Lagrange. An interesting paper by Blitzer[10] demonstrates that minimum potential is not necessary for stability; Blitzer provides several examples of dynamical stability occurring at minimum potential, at maximum potential, at fixed points other than extrema with continuous potential, or even without any potential dependence.

The first attempt to unify stability theory was made by Routh[11]; his essay on stability won for him the Adams prize at Cambridge University. Routh's theorem is not formulated in terms of physical concepts like Lagrange's theorem. Routh showed that the nature of motion of a body is determined by the roots of a certain determinantal equation. He then demonstrated a method of determining stability without finding the roots of the characteristic equation. Rayleigh[12] introduced the dissipation function and investigated the effect of damping on the stability of dynamical systems with holonomic, scleronomic constraints. Kelvin and Tait[13] extended Rayleigh's results to include nonscleronomic, holonomic systems described by linear constant coefficient systems. They discovered the effect of the addition of gyroscopic and damping forces to otherwise stable (or unstable) equilibriums. Chetayev constructed a formal proof using Liapunov's theory; it is now known as the Kelvin-Tait-Chetayev (KTC) theorem. This theorem provides remarkable insight into the physics of the problem.

With the advent of the theory of relativity and quantum mechanics, these results began gathering dust. They were rediscovered by the first generation of spacecraft engineers. Zajac[14] revived the KTC theorem and reformulated the equations in matrix form. He introduced the notion of "pervasive damping" to describe the positive semidefinite damping term appearing in space vehicles. Kane and Barba,[15] in analyzing the stability of a dissipative gravity–oriented spinning satellite, came across a complex set of equations that did not have the structure anticipated by the existing theory. Likins[16] observed that the stiffness matrix is not symmetric in such systems and coined the term "constraint damping" to describe them. Mingori[17] generalized the KTC theorem to deal with systems subject to constraint damping.

For autonomous, holonomic systems, Pringle[18] suggested the use of the Hamiltonian as a Lyapunov function. Unlike earlier methods, which were applicable only to systems with Rayleigh damping, Pringle's results are applicable to systems with arbitrary damping mechanisms.

A recent area of research is the stability of multidimensional second-order linear constant coefficient systems, emerging from the direct application of Newton's or Lagrange's formulation to mechanical systems. Although it is possible to convert the second-order systems into first-order and apply the existing stability criteria, the associated numerical problems force analysts to retain them in second-order. The books by Lancaster and Huseyin, cited in Refs. 19 and 20, cover much of the state of the art in second-order constant coefficient systems and are supplemented with extensive bibliographies.

Frequency Response Methods

An alternate approach to the stability of linear time-invariant systems is through the frequency response methods developed by control engineers. The motivation for the evolution of these methods were 1) the difficulty in factorization of higher-order polynomials and 2) the fact that the solution of the governing differential equation does not provide guidance for improving design. The earliest method available was that of Routh[11] and Hurwitz,[21] by which stability could be ascertained by examining the signs and magnitudes of the coefficients of the characteristic equation without actually having to factor its roots. Liénard and Chipart[22] improved the Routh-Hurwitz criterion by reducing the number of determinantal inequalities by half. The major drawback of the Routh-Hurwitz criterion is its inability to indicate the degree of stability of the system.

The papers of Nyquist,[23] Black,[24] and Bode[25] laid the true foundations for analysis in the frequency domain. The Nyquist criterion uses the frequency response of the open-loop function to predict the stability of the closed-loop system. In contrast to the Routh-Hurwitz criterion, it determines the degree of stability or instability of a feedback control system and aids the designer in improving the response of the system. It can be applied to systems in which no analytic description of the forward and feedback path system functions is available. However it does not provide detailed information concerning the location of the closed-loop system poles as a function of the gain. The Bode diagram, which is basically an extension of the Nyquist criterion, permits a quicker determination of the effects on system response.

The needs of the Second World War strengthened the foundations of feedback control theory and invited a host of talented engineers and mathematicians into its study. The root locus method was introduced by Evans in his two classical papers[26,27] published after the war. The root locus is a plot of the closed-loop poles of the system as a function of the loop gain. Its merits are that 1) a complete, detailed, and accurate solution can be obtained and 2) approximate solutions may be obtained with considerable reduction in labor.

It must be emphasized that, although the frequency response tools were originally developed for feedback systems, they are applicable to any dynamical system.

Periodic Systems

Hill[28,29] and Mathieu[30] initiated research on periodically time-varying systems toward the end of the nineteenth century. There has been hectic activity in this field since then. The writings of Poincaré[31] and Szebehely[32,33] abound with examples of applications of periodic systems. Historical surveys of experimental results and theoretical progress are contained in the classical text by McLachlan[34] and in the recent book by Richards.[35]

The Mathieu equation

$$x'' + (a - 2q \cos 2t)x = 0$$

(a and q are constants) was first encountered in the study of the vibrations of a stretched elliptic membrane. The importance of this equation stems from its appearance in wave motion subject to elliptical boundary conditions.

Floquet's theorem,[36] which is now fundamental in the theory of periodic systems, was proved in 1883. Floquet discovered that any solution of a linear periodic ordinary differential equation may be split into the product of two functions, one periodic and the other exponential. The exponent of the exponential function, called the characteristic exponent, determines the stability of the system. To determine the characteristic exponents, we must numerically compute a fundamental set of solutions during a period of oscillation. This is usually carried out using a computer.

Many important problems in physics can be reduced to the study of a conservative dynamical system with two degrees of freedom. For such systems, if u_n, u_{n+1}, u_{n+2} are the normal displacements of an orbit adjacent to a known periodic orbit in three consecutive revolutions, then the ratio $(u_{n+2} + u_n)/u_{n+1}$, called the stability index,[37] takes a constant value. The orbit is stable or not according to whether the associated stability index is less or greater (in absolute value) than 2. The stability index is related to the characterisitc exponent α and the period T by the equation $k = 2 \cosh \alpha T$.

Three years later, in 1886, Hill, while investigating the motion of lunar perigee, came across the equation

$$x'' + p(t)x = 0$$

where $p(t+\omega) = p(t)$, $\omega \epsilon R$. He developed the infinite determinant method to analyze this equation.

Meissner,[38] in studying instability problems in the side rods of locomotives, was led to an equation of the form

$$x'' + [a - 2q\psi(t)]x = 0$$

where ψ is a rectangular function. This equation can be solved exactly by viewing it as a pair of constant coefficient equations, each valid in alternate time intervals. Van der Pol and Strutt[39] approximated the Mathieu equation by the Meissner equation. Pipes[40] made this approximation a little more sophisticated by taking the ψ in the Meissner equation to be a periodic staircase function and using it to obtain approximate solutions for the Mathieu equation. A number of approximation techniques, running along similar lines, are summarized in Ref. 35.

The damped Mathieu equation

$$x'' + 2\kappa x + (a - 2q \cos 2t)x = 0$$

occurs in a wide variety of practical problems. A recent rediscovery of it appears in Ref. 41, where it is shown that the elastic motion of a large space structure, modeled as a nonspinning axisymmetric beam undergoing flexural vibration, is governed by the damped Mathieu equation. Taylor and Narendra and Gunderson et al.[43] developed analytic criteria for the analysis of this equation. Richards[44,45] used approximation techniques to analyze it. Motivated by developments in large space structures, the authors recently developed an algebraic stability criterion[46] for the damped Mathieu equation.

Starzinskii wrote an exhaustive survey[47] covering the contribution of the Russian school to linear periodic systems. Recent texts by Arscott,[48] D'Angelo,[49] Magnus and Winkler,[50] and Erugin[51] deal exclusively with periodic systems. The aforementioned literature deals mostly with single-degree-of-freedom systems. Yakubovich and Starzinskii[52] present a wide spectrum of methods for the analysis of second-order multidegree-of-freedom systems. Periodically time-varying systems have also been of great interest in spacecraft dynamics. The work in this area is well summarized in the surveys of Shrivastava et al.,[53] Modi,[54] and Shrivastava and Modi.[55]

A related class is that of quasiperiodic or almost periodic systems. Almost periodic solutions and stable solutions are known to be intimately related. In linear time-invariant systems, boundedness of solutions is equivalent to their near periodicity. Bohr[56] and Fink[57] provide interesting reading on almost periodic systems.

Variable Coefficient Equations

Because of its frequent recurrence in practice, a vast amount of literature is available on the single-degree-of-freedom second-order equation

$$x'' + k(t)x = 0$$

Most of the results on this equation and on multidimensional first-order time-varying equations are well documented, with extensive bibliographies in Bellman,[58] Cesari,[59] and Coppell.[60] We summarize the major results.

Several comparison theorems have been proved since the classical theorem of Sturm. Roughly, comparison theorems work in the following way: If the solutions of a differential equation 1 have a property P, then the solutions of a second differential equation 2 are proved to have property P or some related property under some stated connection between 1 and 2. For example, according to one such theorem, if $g(t)$ is a measurable function with

$$\int_0^\infty | g(t) | \, dt < \infty$$

and the solutions of $x'' + f(t)x = 0$ are bounded with their first derivatives, then the same holds for the solutions of $x'' + [f(t) + g(t)]x = 0$. Comparison theorems of this kind have also been derived for L_2 stability.

Lyapunov introduced the concept of reducibility to convert variable coefficient equations to constant coefficient equations, which are more tractable. The system $x' = A(t)x$ is said to be reducible if a matrix $L(t)$ exists whose elements are absolutely continuous functions, which is bounded together with $L^{-1}(t)$ in $[0, \infty)$, and such that $L^{-1}AL - L^{-1}L'$ is a constant matrix in $[0, \infty)$. In such a case, the transformation $x = L(t)y$

reduces $x' = A(t)x$ to a constant coefficient equation. It has been shown that all periodic systems are reducible.

Another concept introduced by Lyapunov is that of characteristic numbers. If A, B are the classes of all real numbers a, b with $f(t)\exp(at)$ bounded, and $f(t)\exp(bt)$ unbounded in $[0, \infty)$, then (A, B) is a partition of the real field defining a real number, say, $-\lambda$. λ is said to be the characteristic number of $f(t)$ in $[0, \infty)$. These numbers are used often.

In attempting to generalize the concept of eigenvalues to variable coefficient matrices, Cesari[61] introduced the concept of generalized characteristic roots. A representative result of Cesari follows. Consider the system $x' = A(t)x$. If $A(t) \rightarrow A_0$ as $t \rightarrow \infty$ and if λ_0 is a simple characteristic root of A_0, then, at large values of t, $A(t)$ has a unique eigenvalue $\lambda(t)$ in the neighborhood of λ_0, and $\lambda(t) \rightarrow \lambda_0$ as $t \rightarrow \infty$. However, this cannot be replaced with the simpler condition that the limit equation $y' = Ay$ be stable.

Many results are available on almost constant coefficient systems, for both second and nth order, i.e., systems of the form $x'' + k(t)x = 0$, where $k(t) \rightarrow a$ as $t \rightarrow \infty$ ($x' = A(t)x$, where $A(t) \rightarrow C$ as $t \rightarrow \infty$). The solutions of these systems have close correspondence with the solutions of $x'' + ax = 0$ ($x' = Cx$). Most theorems prove that, under convenient and more and more general hypotheses, the solutions of $x'' + k(t)x = 0$ ($x' = A(t)x$) behave (or do not behave) as $t \rightarrow \infty$ as the solutions of $x'' + ax = 0$ ($x' = Cx$). The proofs are based on the Bellman-Gronwall lemma.[62,63] The case where $k(t) \rightarrow \infty$ has also been studied extensively. Recently, the authors[64] derived some theorems for almost constant coefficient systems.

The Sonin-Polya theorem,[65] which deals with equations of the form $[p(t)x']' + q(t)x = 0$, establishes the behavior of the solution in terms of the system parameters p and q. This was generalized by Shrivastava[66] to multidegree of freedom systems. In a later paper, Shrivastava and Pradeep[67] applied Lyapunov's theory to the multidimensional equation to derive several theorems.

Contributions of Poincaré and Birkhoff

The nonavailability of explicit solutions for nonlinear systems warrants the use of more sophisticated techniques. The early work in nonlinear systems was confined to devising clever methods for finding solutions to specific equations by "brute force." At the close of the last century, a new era was ushered in by the work of two giants, Poincaré and Lyapunov, who built powerful theories in the analysis of nonlinear systems. These theories still encompass most of our knowledge in the field. These advances were necessitated by Poincaré's discovery of a flaw in the widely accepted proof of Laplace that the solar system was stable. This proof was through series expansion techniques; Poincaré discovered that Laplace erred in concluding that the series was convergent. A little later, Bruns proved that the only quantitative method that could resolve the problem of the stability of the solar system was that of series expansion. Thus was born the necessity of discovery of qualitative methods, which was ingeniously carried out by Poincaré, who brought forth a revolution by his new methods.

Poincaré's fundamental research laid the foundations of topological and analytical[31,68] approaches to nonlinear systems. In initiating the study of topological properties of solutions to second-order ordinary differential equations, Poincaré introduced the concept of a trajectory, which is a parametric curve in the x,x' plane with the parameter t, obtained by eliminating t from the original equation. In this way, a geometric framework was set up to study the qualitative behavior of the solutions.

The following are the salient features of Poincaré's theory of second-order systems: 1) the structure of the trajectories in the entire phase space can be determined by examining the singular points of the system; and 2) an integer, called the index of the singular point, determines the nature of the trajectories around the singular point. The ideas of Poincaré were so far ahead of his time that it took a long time for his contemporaries to understand and assimilate them.

Bendixson[69] clarified some of the questions raised by Poincaré, 15 years after the publication of Poinarcé's classic. One important issue, which was completed by Bendixson, now called the Poincaré-Bendixson theorem, provides sufficient conditions for the existence of periodic solutions of a nonlinear system of differential equations. Roughly, it states that if a closed bounded region M exists containing no equilibrium points such that some trajectory is eventually confined to M, then M contains at least one periodic solution. Although, in practice, it is very difficult to verify that M contains all the limit points of the trajectory, this theorem is marked by its generality and its simple geometric interpretation.

A year before his death, Poincaré[70] announced a conjecture, along with his unsuccessful efforts to prove it. Two years later, Birkhoff provided an elegant proof, which was widely acclaimed. He later extended the original theorem. This line of work has flourished with the work of Kolmogorov, Arnold, and Moser. Birkhoff introduced the abstract notion of dynamical systems; his monograph[72] was the basis for most of the research on dynamical systems. Birkhoff's contributions to the search for periodic solutions are momentous. On this and related matters, Nemytskii's[73] survey paper on topological problems of dynamical systems is excellent. In 1960, Smale,[74] using the concept of structural stability, precisely formulated the fundamental problem of dynamical systems.

An unperturbed two-body motion may be represented as a flow on a torus. When perturbations are applied, the motion can either stay on the perturbed torus or the torus can disappear completely. The Kolmogorov-Arnold-Moser (KAM) theorem, first proposed in 1954 by Kolmogorov[75] and later proved independently by Arnold[76] and Moser,[77], says that under given conditions, most of the tori of the unperturbed system are not destroyed, but only distorted slightly. The theorem, however, leaves open the possibility that resonance phenomena might destroy the tori associated with the unperturbed system.

Poincaré's work is further extended in the theory of bifurcations, in which the effect of changing the value of a parameter on the topological structure of the solutions of a differential equation is studied. There are two basic approaches to the problem: the analytic approach and the algebraic approach. In the analytic approach, which employs the "successor function" of Poincaré, the series expansion of the differential equation is investigated. This method is complicated and has been carried out only for some very special cases. The algebraic approach is much simpler but is applicable only to differential equations that can be reduced to a particular form. Fortunately, this form happens to represent a large class of systems of practical interest. The book by Iooss and Joseph[78] presents the latest results.

The texts by Arnold,[79] Blaquiere,[80] Coddington and Levinson,[81] Cunningham,[82] Hartman,[83] Lefschetz,[84] Minorsky,[85] Sansone and Conti,[86] and Nemytskii and Stepanov[87] contain most of the results of these outstanding men on nonlinear systems.

Lyapunov's Second Method

Almost contemporaneous with Poincaré's discoveries, Lyapunov,[88,89] in his celebrated memoir entitled *The General Problem of Stability of Motion*, laid the foundation of modern stability theory. Whereas Poincaré's singular point analysis is confined to second-order systems, Lyapunov's method is applicable to a much wider class. The underlying idea behind the various theorems of Lyapunov is succinctly summarized by Kalman and Bertram.[90] To quote,

> The principal idea of the second method is contained in the following physical reasoning: If the rate of change $dE(x)/dt$ of the energy $E(x)$ of an isolated physical system is negative for every possible state x, except for a

> single equilibrium state x_e, then the energy will continually decrease until it finally assumes its minimum value $E(x_e)$. In other words, a dissipative system perturbed from its equilibrium will always return to it; this is the intuitive concept of stability.

Lyapunov's second method does not require integration of the variational equations. The stability of the equilibrium point can be inferred from constructing a function (which is named after Lyapunov) that satisfies certain properties. The Lyapunov function has an intuitively appealing geometric interpretation as the energy of the system. Its major setback is that, barring general types of Lyapunov functions that work for certain restricted classes of systems, there is no fixed procedure to construct Lyapunov functions. By and large, construction of Lyapunov functions for new problems is an art.

The term "second method" is due to Lyapunov. The first method, which also finds a place in his memoir, is based on the representation of solutions by infinite series. The second method is also called the "direct method."

One of the most popular techniques for investigating stability is linearization. Under certain conditions, Lyapunov's method enables us to determine the stability of a nonlinear system by using the results obtained from the stability investigation of the linearized system. This is referred to as stability in the first approximation.

Strangely, Lyapunov's methods remained in obscurity for a long time. It was resurrected by Soviet mathematicians about 50 years ago. Chetayev and Malkin were among the first to rediscover Lyapunov's method. They used it for solving problems in aircraft stability analysis, nearly 30 years after the publication of Lyapunov's dissertation. Chetayev and Malkin are mainly responsible for sparking research in this exciting field, in which many Russian scientists were actively involved. The books by Barbashin,[91] Chetayev,[5] Krasovskii,[92] Malkin,[93] and Zubov[94] on Lyapunov's method describe the advances made.

The scientific community of Western Europe and the English-speaking countries remained unaware of the developments in Lyapunov's theory although a French translation of his paper appeared as early as 1907. In 1947, it was translated into English. Antosiewicz[95] published an up-to-date survey of the developments in 1952. It was followed by another survey by Massera[96] in 1956. However, it was not until the publication of the classical papers by Kalman and Bertram[90] and by Parks[97] that the theory gained wide acceptance in the Western world. Texts by Hahn[98,99] and LaSalle and Lefschetz,[100] in addition to the translation of the classical Russian texts mentioned earlier, helped to speed dissemination of the state of art.

There has been considerable interest in converse theorems, which investigate the existence of Lyapunov functions for stable systems. It has been shown that the existence of a Lyapunov function is necessary for various types of stability. Most converse theorems are proved by constructing a suitable auxiliary function. This assumes knowledge of the solution of the differential equation, and therefore converse theorems give no direction to the search for Lyapunov functions. Consequently, they have remained of only theoretical interest.

Integral Manifolds

A useful tool in the analysis of nonlinear time-varying systems is the method of integral manifolds formulated by Bogoliubov[101] in 1945. An integral manifold of a given differential equation is a hypersurface that engulfs the entire solution on its surface, provided that any one value of the solution of the equation is located on it.

The existence of integral manifolds facilitates the study of individual solutions by permitting us to focus attention on the solutions located on the integral manifold instead of studying the entire phase space. This is especially significant for stable integral manifolds where the entire phase space contracts toward the integral manifold. This leads to the reduction in dimensionality of the problem.

It is common practice to approximate complicated equations by simpler, more manageable equations. In many cases, it can be shown that if the approximate equation has a particular integral manifold, then the exact equation also admits an integral manifold located in a small neighborhood of the manifold of the approximate equation.

An initial point taken within an integral manifold generates a different trajectory and hence a new surface. From the property of uniqueness of solutions, it follows that the new integral manifold must lie completely inside the original. Similarly, an initial condition outside of an integral manifold generates a surface outside the original manifold. The region of stability is therefore the largest closed invariant surface that exists.

As applications, extensive numerical generation of invariant manifolds for the stability analysis of planar and coupled librations of gravity-stabilized spacecraft have been carried out by Brereton and Modi[102] and Modi and Shrivastava.[103]

Major Axis Rule and the Energy Sink Method

An established result of classical mechanics is that a rigid body spinning about either the axis of maximum moment of inertia (called the major axis) or the axis of minimum moment of inertia (called the minor axis) in a torque-free environment is stable. Until the late 1950s, it was not known that the small amount of dissipation possessed by every real body makes the spin about the minor axis unstable. This proposition, although not proved rigorously, is now widely accepted and is one of the most significant contributions of spacecraft dynamics to classical mechanics. It has since been known as the "major axis rule."

The major axis rule was first proposed by Landon in 1957. Unfortunately, Landon's theory evoked no response, and his paper was published much later,[104] after being rejected once. Bracewell,[105] on the basis of observations made on Sputnik I, independently discovered it by applying the energy sink argument. Bracewell and Garriot's paper, too, was not published before the launching of the first American satellite, Explorer I. Ironically, it had to be learned afresh for a third time in 1958 by the engineers at the Jet Propulsion Laboratory, who discovered, to their dismay, that the Explorer was tumbling from end to end barely 90 minutes after launch.

The energy sink hypothesis employed by Bracewell was first put forward by Lord Kelvin[12] in his study of the Earth-moon system. This hypothesis postulates that kinetic energy is slowly converted into heat energy during the motion of any body with energy dissipation. The details of the process are not specified, except that it is referred to as an energy sink. Though it must be stressed that the energy sink argument violates Newton's laws of motion in producing changes in motion without applying forces, it provides a good engineering approximation.

Absolute Stability Problem

The absolute stability problem deals with feedback systems whose forward path contains a linear time-invariant subsystem and whose feedback path contains a memoryless time-varying nonlinearity. The main issue, as first formulated by Lur'e and Postnikov[106] in 1944, is to find conditions on the parameters that permit the equilibrium to be globally asymptotically stable, given that the nonlinearity "belongs to the first and third quadrants." In problems of absolute stability, we are concerned not with a particular system but with an entire class of systems. Hence the term "absolute stability problem."

The problem of absolute stability in a finite sector was formulated by Aizerman[107,108] in 1944. Aizerman's conjecture estimates the permissible sector of the nonlinearity for the system to be stable. This conjecture would have turned out to be extremely useful, for it would have enabled us to deduce the stability of the nonlinear system by studying a corresponding linear system. A counterexample constructed by Pliss[109] in 1958 showed this to be false in general. Later, Willems[110] gave a general class of counterexamples to disprove Aizerman's conjecture.

In 1957, Kalman proposed another conjecture, strengthening the hypotheses of Aizerman. This was also shown to be false. The two conjectures of Aizerman and Kalman generated a great deal of interest in the Lur'e problem and attracted the attention of a good number of talented control engineers and mathematicians.

Work on Aizerman's conjecture culminated in Popov's paper,[111] which expressed the conditions for absolute stability in terms of the frequency response of the linear part of the system. Popov's criterion lends itself to graphical interpretation. Popov's results were proved without employing Lyapunov functions. Yakubovich[112] was the first to establish the relationship between his results and Lyapunov functions, which was later refined by Kalman.[113] The relationship is given in the form of a lemma, which shows that, for the existence of the Lur'e type of Lyapunov function, the Popov conditions are both necessary and sufficient. The lemma is known as the Yakubovich-Kalman lemma.

The circle criterion and the Lur'e-Postnikov criterion are two other stability criteria for the Lur'e problem that deserve mention. The circle criterion lends itself to a simple geometric interpretation, like the Nyquist criterion. It is only required to plot the transfer function of the forward path. The Lur'e criterion which is mainly of historic interest, is applicable to a more limited class.

One of the earliest formulations of graphic stability criteria was made by Narendra and Goldwyn.[114] Their results are closely related to the circle criterion.

An extensive literature review on the absolute stability problem is presented by Aizerman and Gantmacher.[115]

Functional Analytic Approach

The functional analytic or input-output approach to the stability of nonlinear time-varying feedback systems relates to the external variables of inputs and outputs. Consequently, the mathematical model takes the form of an operator equation, expressing the relationship between inputs and outputs. The feedback problem is set up in a general framework, with all inputs and outputs taken to be the elements of an extended functional space, enabling us to treat unbounded signals and unstable subsystems. This method was introduced in a series of papers[116–118] independently by Sandberg and Zames. The idea of the input-output method finds its roots in the concept of bounded-input/bounded-output stability and in the work of Nyquist.

There are two basic techniques in the input-output approach: the small gain method and the positivity method. The small gain method, which is similar to the Nyquist criterion, is used to derive the circle criterion for a time-invariant system containing a time-varying, memoryless nonlinearity. The positivity method introduces the idea of using multipliers. Closely related results were obtained using Lyapunov's theory by Brockett and Williams,[119] Yakubovich,[120] Narendra and Neumann,[121] and Anderson.[122]

Desoer[123] showed the Popov's methods could be naturally incorporated into the positivity methods for feedback stability. Zames and Falb[124] placed this on a firm mathematical foundation. Willems and Brockett[125] extend the multiplier criterion to multiple-loop systems. Conditions that insure construction of a multiplier were derived by Cho and Narendra.[126] Modern methods in functional analytic methods are discussed in Desoer and Vidyasagar,[127] Narendra and Taylor,[128] Venkatesh,[129] Vidyasagar,[130] and Willems.[110] In a recent study,[131,132] the authors derived conditions on the L_2 and L_∞ stability of a particular class of first-order systems.

The principal advantage of the functional analysis method is that it covers distributed-parameter and lumped-parameter systems, multi-input/multi-output and single-input/single-output systems, continuous and discrete time systems, all in the same framework.

Chaos

Systems with very sensitive (in fact, exponential) dependence on initial conditions are said to be chaotic. Chaos represents random behavior that is intrinsic to a system and not due to externally imposed noise. Both conservative and dissipative systems can behave chaotically. Examples occur in turbulent fluid motion, vibrating structures, laser-matter interactions, and electronic oscillators. The first observation of chaos was made by Lorenz[133] in his studies of weather prediction. He found that a set of three deterministic nonlinear ordinary differential equations, designed to represent forced dissipative hydrodynamical systems, exhibited for certain parameter ranges the property of very sensitive dependence on initial conditions, a phenomenon now called chaos. What was bewildering was that this random motion resulted from entirely deterministic, simple, innocuous-looking equations. Lorenz arrived at his conclusions, as those engaged in the study of chaos do today, by numerical integration of the equations. Dependence on numerical methods for solution precludes long-term predictions about chaos because, owing to round-off errors, the computer can generate only garbage after a few iterations.

The Lyapunov characteristic exponent (LCE) is a test for chaotic behavior; a system is chaotic if it has at least one positive LCE, and it is orderly if all its exponents are nonpositive. In general, the LCE for a system must be computed numerically. To determine the behavior of the system, it is sufficient to determine the largest characteristic exponent. Most often, however, even this computation is rather lengthy. A computationally cheaper method for distinguishing regular from chaotic motion is Fourier analysis. The power spectrum of chaotic motion is continuous in the frequency domain, as opposed to the power spectrum of regular motion, which appears in the form of sharp spikes.

An estimate of the average time over which accurate predictions can be made about a chaotic system before long-term predictability is lost because of sensitivity to initial conditions can be made through the Kolmogorov entropy. The Kolmogorov entropy K (entropy is the measure of information necessary to determine the state of the system) of a system is defined in such a way that, in most instances, it is equal to the sum of the positive LCE. It represents the average rate of information loss from the system. For nonchaotic systems, as there is no loss of information, K takes the value of zero. For chaotic systems, it takes a positive value.

Chaos is not really as chaotic as may be imagined; a certain order exists in this disorder. It has been found that there are certain universal ways (called routes to chaos) by which a system can make the transition from regular to chaotic behavior. Period doubling (or pitchfork bifurcations) is one such route, in which an attractor (stable fixed point) becomes unstable and gives rise to two attractors as the value of a parameter in the equation is increased. If a system has infinite sequence of period-doubling bifurcations at parameter values λ_1 (first bifurcation, where a stable fixed point becomes unstable and gives rise to two stable fixed points), λ_2 (second bifurcation, where each of the two stable fixed points bifurcate), and so on ad infinitum, then Feigenbaum[134] has established through numerical experiments that the sequence $\{\lambda_n\}$ converges geometrically at a rate $\delta = 4.669$, which is universal.

Another route to chaos is the intermittency route, proposed by Maneville and Pomeau,[135] in which long periods of regular behavior are interruped at random times by short regular bursts. Manneville and Pomeau found that as the parameter r (representing the Rayleigh number) was varied slightly below the critical value r_T in the Lorenz equations, there were regular and stable oscillations for random choice of initial conditions; but as r was increased to a value slightly larger than r_T, there were increasingly frequent, irregular bursts interrupting the regular oscillations, a more or less continuous transition from regular to chaotic behavior. Further, as r was increased, the duration of the laminar phases became shorter and shorter, and the duration of the chaotic phases became longer and longer until, eventually, the motion appeared entirely chaotic. This phenomenon is called intermittency.

Yet another route is the two-frequency route, proposed by Ruelle, Takens, and Newhouse.[136,137] They put forth the argument that a fixed point bifurcates into a periodic orbit, which further bifurcates into a doubly periodic orbit, which then goes directly to chaos under the action of small perturbations without undergoing any more bifurcations.

An oft-used example for studying chaos in Hamiltonian systems is the Henon-Heiles model,[138] which was originally employed for studying the motion of a star in the gravitational field of other stars. The potential was assumed to have cylindrical symmetry; hence, there are two integrals of motion. Through numerical experiments, Henon and Heiles investigated the existence of a third integral of motion, finding that the third integral exists only for a limited range of initial conditions.

Recent books by Sparrow,[139] Steeb and Louw,[140] and Milonni[141] provide excellent introductions to the theory of chaos.

Conclusion

Results on the stability of dynamical systems are reviewed in this paper in a historical perspective. Starting with the concept of stability, the methods of stability analysis in vogue are described. It is seen that the stability of linear time-invariant systems is understood completely. The stability of nonlinear time-invariant systems is not so well known. The recent discovery of chaotic motion is puzzling and offers great challenges to scientists. There is a vast amount of data on chaos from laboratory experiments and from computer simulations that need to be put in order. Study of chaos is rewarding, for a clearer insight into fundamental problems of nature such as turbulence could be obtained from the theory of chaotic motion. In contrast to time-invariant systems, the literature on time-variant systems, both linear and nonlinear, is very scanty. As a consequence of recent trends in science and engineering, time-varying systems are assuming increased importance. In the light of these developments, a fresh look into the stability of time-varying systems is in order. In closing, it may be said that despite the amount of available work, stability theory is still a rich field for research.

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